

Expression for the Number of Spanning Trees of Line Graphs of Arbitrary Connected Graphs*

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Abstract

For any graph G , let $t(G)$ be the number of spanning trees of G , $L(G)$ be the line graph of G and for any non-negative integer r , $S_r(G)$ be the graph obtained from G by replacing each edge e by a path of length $r + 1$ connecting the two ends of e . In this paper we obtain an expression for $t(L(S_r(G)))$ in terms of spanning trees of G by a combinatorial approach. This result generalizes some known results on the relation between $t(L(S_r(G)))$ and $t(G)$ and gives an explicit expression $t(L(S_r(G))) = k^{m+s-n-1}(rk + 2)^{m-n+1}t(G)$ if G is of order $n + s$ and size $m + s$ in which s vertices are of degree 1 and the others are of degree k . Thus we prove a conjecture on $t(L(S_1(G)))$ for such a graph G .

Keywords: Graph; Spanning tree; Line graph; Cayley's Fomula; Subdivision.

1 Introduction

The graphs considered in this article have no loops but may have parallel edges. For any graph G , let $V(G)$ and $E(G)$ be the vertex set and edge set of G respectively, let $S(G)$ be the graph obtained from G by inserting a new vertex to each edge in G , $L(G)$ be the line graph of G , $\mathcal{T}(G)$ be the set of spanning trees of G and $t(G) = |\mathcal{T}(G)|$. Note that for any parallel edges e and e' in G , e and e' are two vertices in $L(G)$ joined by two parallel edges. For any disjoint subsets V_1, V_2 of $V(G)$, let $E_G(V_1, V_2)$ (or simply $E(V_1, V_2)$) denote the set of those edges in $E(G)$ which have ends in V_1 and V_2 respectively, and let $E_G(V_1, V(G) - V_1)$ be simply denoted by $E_G(V_1)$. For any $u \in V(G)$, let $E_G(u)$ (or simply $E(u)$) denote the set $E_G(\{u\})$. So the degree of u in G , denoted by $d_G(u)$ (or simply $d(u)$), is equal to $|E(u)|$. For any subset U of $V(G)$, let $G[U]$ denote the subgraph of G induced by U and let $G - U$ denote

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the subgraph of G induced by $V(G) - U$. For any $E' \subseteq E(G)$, let $G[E']$ be the spanning subgraph of G with edge set E' , $G - E'$ be the graph $G[E(G) - E']$ and G/E' be the graph obtained from G by contracting all edges of E' .

Our paper concerns the relation between $t(G)$ and $t(L(G))$ or $t(L(S(G)))$. Such a relation was first found by Vahovskii [19], then by Kelmans [8] and was rediscovered by Cvetković, Doob and Sachs [7] for regular graphs. They showed that if G is a k -regular graph of order n and size m , then

$$t(L(G)) = k^{m-n-1} 2^{m-n+1} t(G). \quad (1.1)$$

The first result on the relation between $t(G)$ and $t(L(S(G)))$ was found by Zhang, Chen and Chen [21]. They proved that if G is k -regular, then

$$t(L(S(G))) = k^{m-n-1} (k+2)^{m-n+1} t(G). \quad (1.2)$$

Yan [20] recently generalized the result of (1.1). He proved that if G is a graph of order $n+s$ and size $m+s$ in which s vertices are of degree 1 and all others are of degree k , where $k \geq 2$, then

$$t(L(G)) = k^{m+s-n-1} 2^{m-n+1} t(G). \quad (1.3)$$

Yan [20] also proposed a conjecture to generalize the result of (1.2).

Conjecture 1.1 ([20]) *Let G be a connected graph of order $n+s$ and size $m+s$ in which s vertices are of degree 1 and all others are of degree k . Then*

$$t(L(S(G))) = k^{m+s-n-1} (k+2)^{m-n+1} t(G).$$

If G is a digraph, the relation between $t(G)$ and $t(L(G))$ was first obtained by Knuth [9] by an application of the Matrix-Tree Theorem and a bijective proof of the result was found by Bidkhori and Kishore [3]. Note that expressions (1.1), (1.2) and (1.3) were also obtained by the respective authors mentioned above by an application of the Matrix-Tree Theorem. To our knowledge, these results still do not have any combinatorial proofs. Some related results can be seen in [2, 6, 10, 16, 22].

For an arbitrary connected graph G and any non-negative integer r , let $S_r(G)$ denote the graph obtained from G by replacing each edge e of G by a path of length $r+1$ connecting the two ends of e . Thus $S_0(G)$ is G itself and $S_1(G)$ is the graph $S(G)$. Our main purpose in this paper is to use a combinatorial method to find an expression for $t(L(S_r(G)))$ given in Theorem 1.1.

Theorem 1.1 *For any connected graph G and any integer $r \geq 0$,*

$$t(L(S_r(G))) = \prod_{v \in V(G)} d(v)^{d(v)-2} \sum_{E' \subseteq E(G)} t(G[E']) r^{|E'| - |V(G)|+1} \prod_{e \in E(G) - E'} (d(u_e)^{-1} + d(v_e)^{-1}), \quad (1.4)$$

where $d(v) = d_G(v)$ and u_e and v_e are the two ends of e .

As $S_0(G)$ is G itself, the following expression for $t(L(G))$ is a special case of Theorem 1.1:

$$t(L(G)) = \prod_{v \in V(G)} d(v)^{d(v)-2} \sum_{T \subseteq \mathcal{T}(G)} \prod_{e \in E(G)-E(T)} (d(u_e)^{-1} + d(v_e)^{-1}). \quad (1.5)$$

The proof of Theorem 1.1 will be completed in Sections 3 and 4. In Section 3, we will show that the case $r = 0$ of Theorem 1.1 (i.e., the result (1.5)) is a special case of another result (i.e., Theorem 3.1), and in Section 4, we will prove the case $r \geq 1$ of Theorem 1.1 by applying this theorem for the case $r = 0$ (i.e., (1.5)). To establish Theorem 3.1, we need to apply a result in Section 2 (i.e., Proposition 2.3), which determines the number of spanning trees in a graph G with a clique V_0 such that $F = G - E(G[V_0])$ is a forest and every vertex in V_0 is incident with at most one edge in F . Finally, in Section 5, we will apply Theorem 1.1 to show that for any graph G mentioned in Conjecture 1.1 and any integer $r \geq 0$, we have

$$t(L(S_r(G))) = k^{m+s-n-1} (rk + 2)^{m-n+1} t(G). \quad (1.6)$$

Thus (1.3) follows and Conjecture 1.1 is proved.

Note that in the proof of Theorem 1.1, we will express $t(L(S_r(G)))$ in another form (i.e., (1.8)), which is actually equivalent to (1.4).

For any graph G and any $E' \subseteq E(G)$, let $\Gamma(E')$ be the set of those mappings $g : E' \rightarrow V(G)$ such that for each $e \in E'$, $g(e) \in \{u_e, v_e\}$, where u_e and v_e are the two ends of e . Observe that

$$\sum_{g \in \Gamma(E')} \prod_{v \in V(G)} d(v)^{-|g^{-1}(v)|} = \prod_{e \in E'} (d(u_e)^{-1} + d(v_e)^{-1}). \quad (1.7)$$

Thus (1.4) and (1.5) can be replaced by the following expressions:

$$t(L(S_r(G))) = \sum_{E' \subseteq E(G)} t(G[E']) r^{|E'| - |V(G)|+1} \sum_{g \in \Gamma(E(G)-E')} \prod_{v \in V(G)} d(v)^{d(v)-2-|g^{-1}(v)|} \quad (1.8)$$

and

$$t(L(G)) = \sum_{T \in \mathcal{T}(G)} \sum_{g \in \Gamma(E(G)-E(T))} \prod_{v \in V(G)} d(v)^{d(v)-2-|g^{-1}(v)|}. \quad (1.9)$$

2 Preliminary Results

In this section, we shall establish some results which will be used in the next section to prove Theorem 1.1 for the case $r = 0$.

For any connected graph H and any forest F of H , let $\mathcal{ST}_H(F)$ be the set of those spanning trees of H containing all edges of F , and $\mathcal{SF}_H(F)$ be the set of those spanning forests of H containing all edges of F .

In this section, we always assume that G is a connected graph with a clique V_0 such that $F = G - E(G[V_0])$ is a forest and every vertex of V_0 is incident with at most one edge of F , as shown in Figure 1.

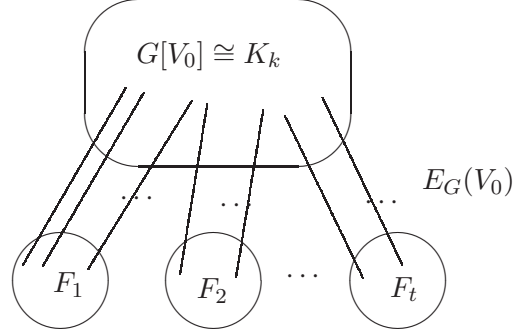


Figure 1: V_0 is a clique of G such that and $G - E(G[V_0])$ is a forest

Let $k = |V_0|$, $d = |E_G(V_0)|$, $t = c(G - V_0)$ and F_1, F_2, \dots, F_t be components of $G - V_0$. Observe that $k \geq d \geq t$, as $|E_G(v, V - V_0)| \leq 1$ holds for each $v \in V_0$ and $|E_G(V_0, V(F_i))| \geq 1$ holds for each F_i .

The main purpose in this section is to show that if $k > d$, then the set $\mathcal{ST}_G(F)$ can be equally partitioned into $\prod_{1 \leq j \leq t} |E_G(V_0, V(F_j))|$ subsets, each of which has its size $k^{k-2+t-d}$.

In the following, we divide this section into two parts.

2.1 A preliminary result on trees

In this subsection, we shall establish some results on trees which are needed for the next subsection and following sections.

Let T be any tree and V_0 be any proper subset of $V(T)$. Observe that identifying all vertices in V_0 changes T to a connected graph which is a tree if and only if $|E_T(V_0)| = c(T - V_0)$. So the following observation is obvious.

Lemma 2.1 *Let $t = c(T - V_0)$ and S be any proper subset of $E_T(V_0)$. Then the two statements below are equivalent:*

- (i) $|S \cap E_T(V_0, V(F_i))| = 1$ holds for all components F_1, F_2, \dots, F_t of $T - V_0$;
- (ii) the graph obtained from T by removing all edges in the set $E_T(V_0) - S$ and identifying all vertices of V_0 is a tree.

With T, V_0 given above together with a special vertex $v \in V_0$ such that $N(v) \subseteq V_0$, a subset S of $E_T(V_0)$ with the properties in Lemma 2.1 will be determined by a procedure below (i.e.,

Algorithm A). As S is uniquely determined by T, V_0 and v , we can denote it by $\Phi(T, V_0, v)$. Thus $|\Phi(T, V_0, v)| = t = c(T - V_0)$.

Roughly, if $t = 1$, the only edge of $\Phi(T, V_0, v)$ will be selected from $E_T(V_0)$ according to the condition that it has one end in the same component of $T[V_0]$ as v ; if $t \geq 2$, the t edges of $\Phi(T, V_0, v)$ will be determined by the $t - 1$ paths P_2, P_3, \dots, P_t in T , where P_j is the shortest path connecting vertices of F_1 and vertices of F_j for $j = 2, 3, \dots, t$ and F_1, F_2, \dots, F_t are the components of $T - V_0$.

Assume that in Algorithm A, $E(T) = \{e_i : i \in I\}$ for some finite I of positive integers.

Algorithm A with input (T, V_0, v) :

Step A1. Let $t = c(T - V_0)$.

Step A2. If $t = 1$, let $\Phi = \{e_j\}$, where e_j is the unique edge in the set $E_T(V_0)$ which has one end in the component of $T[V_0]$ containing v . Go to Step A5.

Step A3. (Now we have $t \geq 2$.)

A3-1. The components of $T - V_0$ are labeled as F_1, F_2, \dots, F_t such that

$$\min\{s : e_s \in E_T(V_0, F_i)\} < \min\{s' : e_{s'} \in E_T(V_0, F_{i+1})\} \quad (2.1)$$

for all $i = 1, 2, \dots, t - 1$. (In other words, these components are sorted by the minimum edge labels. For example, for the tree T in Figure 2(a), the four components F_1, F_2, F_3, F_4 of $T - V_0$ are labeled according to this rule.)

A3-2. For $j = 2, 3, \dots, t$, determine the unique path P_j in T which is the shortest one among all those paths in T connecting vertices of F_1 to vertices of F_j .

Step A4. Let $\Phi = (E(P_2) \cap E_T(V_0, V(F_1))) \cup \bigcup_{j=2}^t (E(P_j) \cap E_T(V_0, V(F_j)))$.

Step A5. Output Φ .

Remarks:

- (i) Vertex v is needed only for the case that $t = 1$;
- (ii) If $t = 1$, the only edge of Φ is uniquely determined as T is a tree and $T - V_0$ is connected;
- (iii) As T is a tree and F_1 and F_j are connected, P_j is actually the only path of T with its ends in F_1 and F_j respectively and every internal vertex of P_j does not belong to $V(F_1) \cup V(F_j)$. Thus for P_j chosen in Step A3,

$$|E(P_j) \cap E_G(V_0, V(F_1))| = |E(P_j) \cap E_G(V_0, V(F_j))| = 1,$$

implying that by Step A4, $|\Phi \cap E_G(V_0, V(F_j))| = 1$ for all $j = 1, 2, \dots, t$.

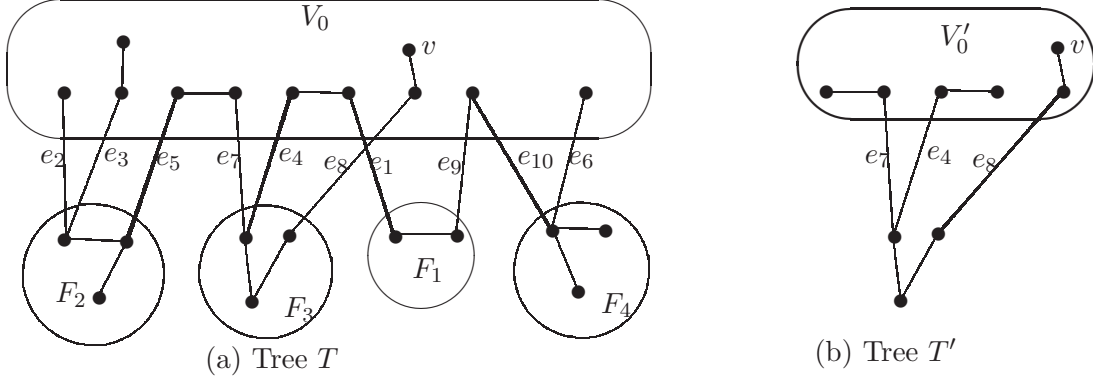


Figure 2: $\Phi(T, V_0, v) = \{e_1, e_4, e_5, e_{10}\}$ and $\Phi(T', V'_0, v) = \{e_8\}$

For example, for the tree T with V_0 and v shown in Figure 2 (a), $T - V_0$ has four components F_1, F_2, F_3, F_4 labeled according to the minimum edge labels, and running Algorithm A with input (T, V_0, v) gives $\Phi(T, V_0, v) = \{e_1, e_4, e_5, e_{10}\}$, as the three paths P_2, P_3 and P_4 obtained by the algorithm have properties that $\{e_1, e_5\} \subseteq E(P_2)$, $\{e_1, e_4\} \subseteq E(P_3)$ and $\{e_9, e_{10}\} \subseteq E(P_4)$. For the tree T' in Figure 2 (b), $T' - V'_0$ has one component only and $\Phi(T', V'_0, v) = \{e_8\}$. Note that vertex v is used for finding $\Phi(T', V'_0, v)$ but not for finding $\Phi(T, V_0, v)$.

Our second purpose in this subsection is to show that if $|E_T(V_0, V(F_j))| > 1$ for some component F_j of $T - V_0$, we can find another tree T' with $V(T') = V(T)$ and $T' - E(T'[V_0]) = T - E(T[V_0])$ such that $\Phi(T', V_0, v)$ and $\Phi(T, V_0, v)$ are different only at choosing the edge joining a vertex of V_0 to a vertex in F_j .

For two distinct edges e, e' of $E_T(V_0)$ incident with u and u' respectively, where $u, u' \in V_0$, let $T(e \leftrightarrow e')$ be the graph, as shown in Figure 3, obtained from T by changing every edge (u, w) of $T[V_0]$, where $w \neq u'$, to (u', w) and every edge (u', w') of $T[V_0]$, where $w' \neq u$, to (u, w') .

Roughly, $T(e \leftrightarrow e')$ is actually obtained from T by exchanging $(N_T(u) \cap V_0) - \{u'\}$ with $(N_T(u') \cap V_0) - \{u\}$. Note that u and u' are adjacent in T if and only if they are adjacent in $T(e \leftrightarrow e')$.

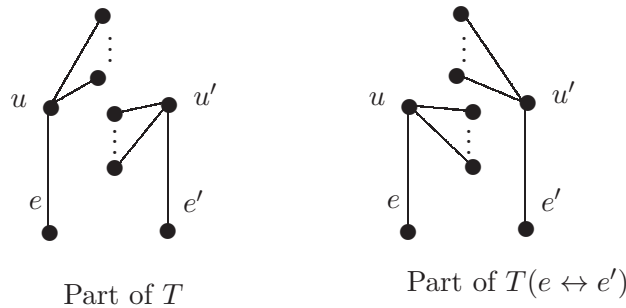


Figure 3: T and $T(e \leftrightarrow e')$

Let T' denote $T(e \leftrightarrow e')$ in the remainder of this subsection. There is a bijection $\tau : E(T) \rightarrow$

$E(T')$ defined below: $\tau(e) = e'$, $\tau(e') = e$, $\tau((u, w)) = (u', w)$ whenever $(u, w) \in E(T)$ for $w \in V_0 - \{u'\}$, $\tau((u', w')) = (u, w')$ whenever $(u', w') \in E(T)$ for $w' \in V_0 - \{u\}$, and $\tau(e'') = e''$ for all other edges e'' in T .

Note that T' may be not a tree, although $T' - V_0$ and $T - V_0$ are the same graph and F_1, F_2, \dots, F_t are also the components of $T' - V_0$. But T' is indeed a tree if both e and e' have ends in the same component of $T - V_0$.

Lemma 2.2 *Let e, e' be distinct edges of $E_T(V_0, V(F_i))$ for some i with $1 \leq i \leq t$.*

- (i) *Then T' is a tree;*
- (ii) *If $e \in \Phi(T, V_0, v)$ and either $t \geq 2$ or $N_T(v) \subseteq V_0$, then $\Phi(T', V_0, v) = (\Phi(T, V_0, v) - \{e\}) \cup \{e'\}$.*

Proof. Note that for any edge $e'' \in E(T - V_0)$, T/e'' is also a tree, T' is a tree if and T'/e'' is a tree, and more importantly, $\Phi(T, V_0, v) = \Phi(T/e'', V_0, v)$. Thus it suffices to prove this lemma only for the case that $|V(F_i)| = 1$ for all $i = 1, 2, \dots, t$.

- (i) It can be proved easily by induction on the number of edges in T .
- (ii) Assume that $t = 1$. Then $N_T(v) \subseteq V_0$ and so v is not any end of e . As $e \in \Phi(T, V_0, v)$, $\Phi(T, V_0, v) = \{e\}$. By Algorithm A, e has one end (i.e., u) in the component of $T[V_0]$ containing v (i.e., the subgraph $T[V_0]$ has a path P connecting v to u). By the definition of T' (i.e., $T(e \leftrightarrow e')$), P is now changed to a path P' in $T'[V_0]$ by the mapping τ connecting v to one end of e' (i.e., u'). Thus $\Phi(T', V_0, v) = \{e'\}$ by Algorithm A. The result holds for this case.

Now assume that $t \geq 2$. For $j = 2, 3, \dots, t$, let P_j be the only path in T with its two ends in F_1 and F_j respectively and every interval vertex of P_j does not belong to $V(F_1) \cup V(F_j)$.

With the bijection $\tau : E(T) \rightarrow E(T')$ defined above, for $j = 2, 3, \dots, t$, $\tau(E(P_j))$ is a subset of $E(T')$ and forms a path in T' , denoted by P'_j . Note that the two ends of P'_j are in F_1 and F_j respectively and every interval vertex of P'_j does not belong to $V(F_1) \cup V(F_j)$. Also observe that for $j = 2, 3, \dots, t$, if $i \in \{1, j\}$, then

$$E(P'_j) \cap E_{T'}(V_0, V(F_i)) = \{e'\},$$

and if $s \in \{1, j\} - \{i\}$, then

$$E(P'_j) \cap E_{T'}(V_0, V(F_s)) = E(P_j) \cap E_T(V_0, V(F_s)).$$

Hence (ii) holds. □

2.2 Partitions of $\mathcal{ST}_G(F)$

Recall that G is a connected graph with a clique V_0 of order k such that $F = G - E(G[V_0])$ is a forest and every vertex of V_0 is incident with at most one edge of F (i.e., $d_F(v) \leq 1$ for each $v \in V_0$), as shown in Figure 1. In this subsection, our main purpose is to partition $\mathcal{ST}_G(F)$ equally into $\prod_{j=1}^t |E_G(V_0, V(F_j))|$ subsets, where F_1, F_2, \dots, F_t are the components of $G - V_0$.

We start with the following beautiful formula for the number of spanning trees of a complete graph K_k of order k containing a given spanning forest. This result was originally due to Lovász (Problem 4 in page 29 of [11]).

Theorem 2.1 ([11]) *For any spanning forest F of K_k , if c is the number of components of F and k_1, k_2, \dots, k_c are the orders of its components, then*

$$|\mathcal{ST}_{K_k}(F)| = k^{c-2} \prod_{i=1}^c k_i.$$

This result naturally generalizes the well-known formula that $t(K_k) = k^{k-2}$ for any $k \geq 1$, which was first obtained by Cayley [1]. Now we apply this result to establish some results on the set $\mathcal{ST}_G(F)$ and finally partition $\mathcal{ST}_G(F)$ equally into $\prod_{j=1}^t |E_G(V_0, V(F_j))|$ subsets.

Recall that $d = |E_G(V_0)|$ and $k \geq d \geq t$.

Proposition 2.1 *With G, F and V_0 defined above, we have*

$$|\mathcal{ST}_G(F)| = k^{k-2+t-d} \prod_{j=1}^t |E_G(V_0, V(F_j))|.$$

Proof. We only need to consider the case that $E_G(V_0, V(F_j)) \neq \emptyset$ for every component F_j of $G - V_0$; otherwise, the result is trivial as $|\mathcal{ST}_G(F)| = 0$ when G is disconnected.

Observe that for any edge e of $E(G - V_0)$, we have $|\mathcal{ST}_{G/e}(F/e)| = |\mathcal{ST}_G(F)|$. Thus we may assume that every component of $G - V_0$ is a single vertex, implying that $G - V_0$ is the empty graph of t vertices, namely x_1, x_2, \dots, x_t . So $E(F) = E_G(V_0)$.

For each $j = 1, 2, \dots, t$, let $V_j = \{x \in V_0 : x \text{ is incident with } x_j\}$ and e_j be any edge joining x_j to some vertex in V_j . Let $G_0 = G[V_0]$. Note that $F/\{e_1, e_2, \dots, e_t\}$ can be considered as a spanning forest of G_0 and

$$\mathcal{ST}_G(F) = \mathcal{ST}_{G_0}(F/\{e_1, e_2, \dots, e_t\}).$$

As G_0 is a complete graph of order k , by Theorem 2.1,

$$|\mathcal{ST}_{G_0}(F/\{e_1, e_2, \dots, e_t\})| = k^{c-2} \prod_{j=1}^c |V'_j|,$$

where c is the number of components of $F/\{e_1, e_2, \dots, e_t\}$ and V'_1, V'_2, \dots, V'_c are vertex sets of components of $F/\{e_1, e_2, \dots, e_t\}$. Note that

$$|V_0 - \bigcup_{j=1}^t V_j| = |V_0| - \sum_{k=1}^t |V_j| = k - |E_G(V_0)| = k - d,$$

implying that $c = k - d + t$ and the sizes of V'_1, V'_2, \dots, V'_c are equal to

$$|V_1|, \dots, |V_t|, \underbrace{1, \dots, 1}_{k-d}.$$

As $|V_j| = |E_G(V_0, \{x_j\})| = |E_G(V_0, V(F_j))|$, the result follows from Theorem 2.1. \square

Now assume that v is a vertex of V_0 with $N_G(v) \subseteq V_0$, i.e., $d_F(v) = 0$. Note that this condition is only needed for the case that $G - V_0$ is connected. Under this condition, it is obvious that $k > d$.

Recall that for any tree T of $\mathcal{ST}_G(F)$, $\Phi(T, V_0, v)$ is a subset of $E_G(V_0)$ with the property that $|\Phi(T, V_0, v) \cap E_G(V_0, V(F_j))| = 1$ for each $j = 1, 2, \dots, t$. For each subset S of $E_G(V_0)$ with the property that $|S \cap E_G(V_0, V(F_j))| = 1$ for each $j = 1, 2, \dots, t$, let $\mathcal{ST}_G(F, S, v)$ denote the set of those spanning trees $T \in \mathcal{ST}_G(F)$ with $\Phi(T, V_0, v) = S$. Thus $\mathcal{ST}_G(F)$ is partitioned into $\prod_{j=1}^t |E_G(V_0, V(F_j))|$ subsets $\mathcal{ST}_G(F, S, v)$'s. The following result shows that all these sets $\mathcal{ST}_G(F, S, v)$'s have the same size.

The following result shows that $|\mathcal{ST}_G(F, S, v)|$ is independent of S .

Proposition 2.2 *Assume that $k > d$ and $N(v) \subseteq V_0$. For any subset S of $E_G(V_0)$ with $|S \cap E_G(V_0, V(F_j))| = 1$ for each component F_j of $G - V_0$, we have*

$$|\mathcal{ST}_G(F, S, v)| = k^{k-2+t-d}.$$

Proof. There are exactly $\prod_{j=1}^t |E_G(V_0, V(F_j))|$ subsets S of $E_G(V_0)$ with the property that $|S \cap E_G(V_0, V(F_j))| = 1$ for each component F_j of $G - V_0$. By Proposition 2.1, we only need to show that $|\mathcal{ST}_G(F, S, v)| = |\mathcal{ST}_G(F, S', v)|$ holds for any two such sets S and S' . Thus it suffices to show that $|\mathcal{ST}_G(F, S, v)| = |\mathcal{ST}_G(F, S', v)|$ holds for any two such sets S and S' with $|S - S'| = 1$, i.e., S and S' have exactly $t - 1$ same edges.

Let S be such a subset of $E_G(V_0)$ mentioned above. Assume that e, e' are distinct edges in $E_G(V_0, V(F_j))$ for some j with $e \in S$ and $e' \notin S$. Let $S' = (S - \{e\}) \cup \{e'\}$. It remains to show that $|\mathcal{ST}_G(F, S, v)| = |\mathcal{ST}_G(F, S', v)|$.

For any $T \in \mathcal{ST}_G(F, S, v)$, let T' be the tree $T(e \leftrightarrow e')$. By Lemma 2.2, we have $\Phi(T', V_0, v) = (\Phi(T, V_0, v) - \{e\}) \cup \{e'\}$, implying that $T' \in \mathcal{ST}_G(F, S', v)$.

Let ϕ be the mapping from $\mathcal{ST}_G(F, S, v)$ to $\mathcal{ST}_G(F, S', v)$ defined by $\phi(T) = T(e \leftrightarrow e')$. It is obvious that ϕ is an onto mapping, and $\phi' : T' \rightarrow T'(e' \leftrightarrow e)$ is also an onto mapping from $\mathcal{ST}_G(F, S', v)$ to $\mathcal{ST}_G(F, S, v)$.

Thus $|\mathcal{ST}_G(F, S, v)| = |\mathcal{ST}_G(F, S', v)|$ and the result follows. \square

We end this section with an application of Proposition 2.2 to deduce another result.

Let G' be the graph obtained from G by contracting all edges in $G[V_0]$. Then V_0 becomes a vertex in G' , denoted by v_0 . Thus $V(G') = (V(G) - V_0) \cup \{v_0\}$, and $E(G')$ and $E(G) - E(G[V_0])$ are the same although for each edge $e \in E_G(V_0)$, its end in V_0 is changed to v_0 when e becomes an edge in G' . An example for G and G' is shown in Figure 4 (a) and (b).

Let T' be any spanning tree of G' with $E(G' - v_0) \subseteq E(T')$, i.e., $T' \in \mathcal{ST}_{G'}(F')$ for $F' = G' - v_0$. Thus $|E_{T'}(v_0)| = t$, i.e., $E_{T'}(v_0)$ has exactly t edges, corresponding to t edges in G , one from $E_G(V_0, V(F_j))$ for each component F_j of $G - V_0$. An example for T' is shown in Figure 4 (d).

Let D be any subset of $E_G(V_0) - E_{T'}(v_0)$ and let $\mathcal{ST}_G(V_0, T', D, v)$ be the set of those spanning trees T of G such that (i) $T - V_0$ and $T' - v_0$ are the same graph; (ii) $E_T(V_0)$ is the disjoint union of D and $E_{T'}(v_0)$ and (iii) $\Phi(T, V_0, v) = E_{T'}(v_0)$. Thus $\mathcal{ST}_G(V_0, T', D, v) \subseteq \mathcal{ST}_G(F)$ if and only if $D = E_G(V_0) - E_{T'}(v_0)$. For example, the tree T in Figure 4 (c) belongs to $\mathcal{ST}_G(V_0, T', D, v)$ with $D = \{e_1, e_5\}$, but $T \notin \mathcal{ST}_G(F)$, as $E(F) \not\subseteq E(T)$.

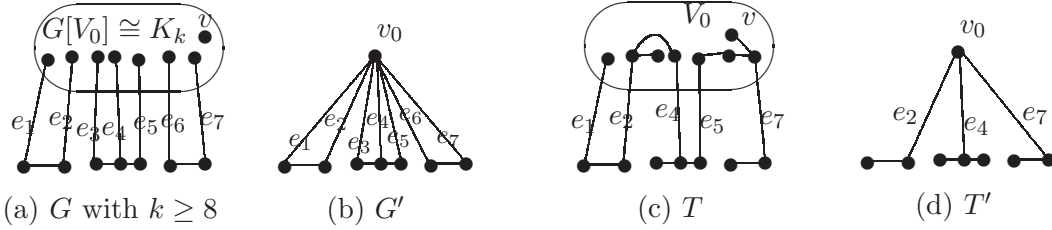


Figure 4: A tree T in $\mathcal{ST}_G(V_0, T', D, v)$ with $D = \{e_1, e_5\}$

Proposition 2.3 *With T' and D given above, we have*

$$|\mathcal{ST}_G(V_0, T', D, v)| = k^{k-2-|D|}.$$

Proof. Let G^* denote the graph $G - D'$, where $D' = E_G(V_0) - (D \cup E_{T'}(v_0))$. Observe that

$$\mathcal{ST}_G(V_0, T', D, v) = \mathcal{ST}_{G^*}(F^*, E_{T'}(v_0), v),$$

where $F^* = G^* - E(G^*[V_0])$, i.e., $F^* = F - D'$. Also note that $c(G^* - V_0) = c(G - V_0) = t$ and

$$|E_{G^*}(V_0)| = |E_{T'}(v_0) \cup D| = t + |D|.$$

By Proposition 2.2, we have

$$|\mathcal{ST}_{G^*}(F^*, E_{T'}(v_0), v)| = k^{k-2+t-(t+|D|)} = k^{k-2-|D|}.$$

Thus the result holds. \square

3 Proving Theorem 1.1 for $r = 0$

In this section, we shall prove Theorem 1.1 for the case $r = 0$ (i.e., the result of (1.9) or equivalently (1.5)) is a special case of another result (i.e., Theorem 3.1).

Let u be any vertex in a simple graph G . Assume that $E_G(u) = \{(u, u_i) : 1 \leq i \leq s\}$, where $s = d_G(u)$. If G' is the graph obtained from $G - u$ by adding a complete graph K_s with vertices w_1, w_2, \dots, w_s and adding s new edges (w_i, u_i) for $i = 1, 2, \dots, s$, then G' is said to be obtained from G by a *clique-insertion at u* . The clique-insertion is a graph operation playing an important role in the study of vertex-transitive graphs (see [12, 14]). The *clique-inserted graph* of G , denoted by $C(G)$, is obtained from G by operating clique-insertion at every vertex of G . Note that the clique-inserted graph of G is also called *the para-line graph* of G (see [18]). An example for $C(G)$ is shown in Figure 5.

Let M be the set of those edges in $E(C(G))$ which are not in the inserted cliques. So M consists of all edges in $E(G)$ and thus can be considered as the same as $E(G)$. Observe that $C(G)$ has the following properties:

- (i) M is a matching of $C(G)$;
- (ii) $L(G)$ is the graph $C(G)/M$ and thus $t(L(G)) = |\mathcal{ST}_{C(G)}(M)|$;
- (iii) each component of $C(G) - M$ is a complete graph.

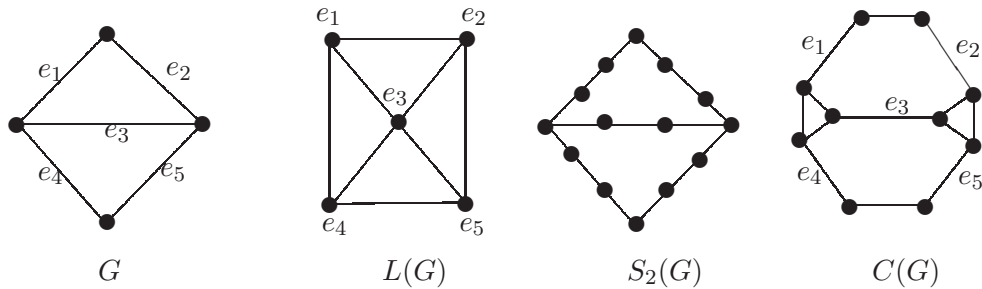


Figure 5: Line graph $L(G)$ and clique-inserted graph $C(G)$

From observation (iii) above, $C(G)$ is in a type of connected graphs with a matching whose removal yields components which are all complete graphs. As $t(L(G)) = |\mathcal{ST}_{C(G)}(M)|$ holds for any connected graph G with M defined above, we now extend our problem to finding an expression for $|\mathcal{ST}_Q(M)|$, where Q is an arbitrary connected graph and M is any matching of Q such that all components of $Q - M$ are complete graphs.

Throughout this section, we assume

- (i) Q is a simple and connected graph with a matching M such that all components Q_1, Q_2, \dots, Q_n of $Q - M$ are complete graphs;

- (ii) for $i = 1, 2, \dots, n$, $V_i = V(Q_i) = \{v_{i,j} : j = 1, 2, \dots, k_i\}$, where $k_i = |V_i|$;
- (iii) $M = \{e_1, e_2, \dots, e_m\}$ and M_i is the set of those edges of M which have one end in V_i and $m_i = |M_i|$ for $i = 1, 2, \dots, n$;
- (iv) $v_{i,j}$ is incident with an edge of M_i if and only if $1 \leq j \leq m_i$;
- (v) Q^* is the graph obtained from Q by contracting all edges of Q_i for all $i = 1, 2, \dots, n$.
Thus each Q_i is converted to a vertex in Q^* denoted by v_i .

With the above assumptions, we observe that $V(Q^*) = \{v_1, v_2, \dots, v_n\}$ and $E(Q^*) = M$. As M is a matching of Q and Q is connected, we have $1 \leq m_i \leq k_i$. If $k_i > m_i$, then vertex $v_{i,j}$ is not incident with any edge of M for all $j : m_i < j \leq k_i$. If $k_i = m_i$ for all $i = 1, 2, \dots, n$, then $|\mathcal{ST}_Q(M)| = t(L(Q^*))$. Thus result (1.9) is a special case of Theorem 3.1 which is the main result to be established in this section.

Theorem 3.1 *For Q, Q^* and M defined above, we have*

$$|\mathcal{ST}_Q(M)| = \sum_{T \in \mathcal{T}(Q^*)} \sum_{f \in \Gamma(E(Q^*) - E(T))} \prod_{i=1}^n k_i^{k_i - 2 - |f^{-1}(v_i)|}. \quad (3.1)$$

To prove Theorem 3.1, by the following result, we only need to consider the case that $k_i > m_i$ for all $i = 1, 2, \dots, n$.

Proposition 3.1 *Theorem 3.1 holds if it holds whenever $k_i > m_i$ for all $i = 1, 2, \dots, n$.*

Proof. Assume that M is fixed and so all m_i 's are fixed. Without loss of generality, we only need to show that with k_i , where $k_i \geq m_i$, to be fixed for all $i = 2, \dots, n$, if (3.1) holds for every integer k_1 with $k_1 \geq m_1 + 1$, then it also holds for the case $k_1 = m_1$.

For any integer $k_1 \geq m_1$, let

$$\gamma(k_1) = |\mathcal{ST}_Q(M)|.$$

By the assumption, for any $k_1 \geq m_1 + 1$, (3.1) holds and thus

$$\gamma(k_1) = \sum_{T \in \mathcal{T}(Q^*)} \sum_{f \in \Gamma(E(Q^*) - E(T))} k_1^{k_1 - 2 - |f^{-1}(v_1)|} \prod_{i=2}^n k_i^{k_i - 2 - |f^{-1}(v_i)|} = \sum_{s=0}^{m_1-1} a_s k_1^{k_1 - 2 - s}, \quad (3.2)$$

where

$$a_s = \sum_{T \in \mathcal{T}(Q^*)} \sum_{\substack{f \in \Gamma(E(Q^*) - E(T)) \\ |f^{-1}(v_1)| = s}} \prod_{i=2}^n k_i^{k_i - 2 - |f^{-1}(v_i)|}. \quad (3.3)$$

It is clear that a_s is independent of the value of k_1 .

Now let Q' be the graph $Q - E(Q_1) - \{v_{1,j} : m_1 < j \leq k_1\}$. So Q' is independent of k_1 . Note that for every $T \in \mathcal{ST}_Q(M)$, $F = T - E(T[V_1]) - \{v_{1,j} : m_1 < j \leq k_1\}$ is a member of $\mathcal{SF}_{Q'}(M)$, i.e., a spanning forest of Q' containing all edges of M , since $v_{1,j}$ is not incident with any edge of M for all $j : m_1 < j \leq k_1$. Thus $\mathcal{ST}_Q(M)$ can be partitioned into

$$\mathcal{ST}_Q(M) = \bigcup_{F \in \mathcal{SF}_{Q'}(M)} \mathcal{ST}_{Q''}(F),$$

where $Q'' = Q[E(F) \cup E(Q_1)]$. It is possible that $\mathcal{ST}_{Q''}(F) = \emptyset$ for some $F \in \mathcal{SF}_{Q'}(M)$. But $\mathcal{ST}_{Q''}(F') \cap \mathcal{ST}_{Q''}(F'') = \emptyset$ for distinct $F', F'' \in \mathcal{SF}_{Q'}(M)$, implying that for any $k_1 = |V_1| \geq m_1$,

$$\gamma(k_1) = \sum_{F \in \mathcal{SF}_{Q'}(M)} |\mathcal{ST}_{Q''}(F)|.$$

By Proposition 2.1, for any $F \in \mathcal{SF}_{Q'}(M)$, if $F/\{v_{1,j} : 1 \leq j \leq m_1\}$ is connected, then

$$|\mathcal{ST}_{Q''}(F)| = k_1^{k_1-2+c(F-V_1)-m_1} \prod_{j=1}^{c(F-V_1)} |E_F(V_1, V(F_j))|,$$

where $F_1, F_2, \dots, F_{c(F-V_1)}$ are the components of $F - V_1$. Let $\mathcal{SF}_{Q'}^c(M)$ denote the set of those $F \in \mathcal{SF}_{Q'}(M)$ such that $F/\{v_{1,j} : 1 \leq j \leq m_1\}$ is connected. Thus, for any $k_1 \geq m_1$, we have

$$\begin{aligned} \gamma(k_1) &= \sum_{F \in \mathcal{SF}_{Q'}^c(M)} k_1^{k_1-2+c(F-V_1)-m_1} \prod_{j=1}^{c(F-V_1)} |E_F(V_1, V(F_j))| \\ &= \sum_{s=0}^{m_1-1} b_s k_1^{k_1-2-s}, \end{aligned} \quad (3.4)$$

where

$$b_s = \sum_{\substack{F \in \mathcal{SF}_{Q'}^c(M) \\ c(F-V_1)=m_1-s}} \prod_{j=1}^{c(F-V_1)} |E_F(V_1, V(F_j))|. \quad (3.5)$$

As Q' is independent of k_1 , for any $F \in \mathcal{SF}_{Q'}^c(M)$, the expression $\prod_{j=1}^{c(F-V_1)} |E_F(V_1, V(F_j))|$ is independent of $k_1 = |V_1|$ and hence b_s is independent of k_1 .

By (3.2) and (3.4), for every integer k_1 with $k_1 \geq m_1 + 1$, we have

$$\sum_{s=0}^{m_1-1} a_s k_1^{k_1-2-s} = \sum_{s=0}^{m_1-1} b_s k_1^{k_1-2-s}, \quad (3.6)$$

where a_s and b_s are independent of k_1 for all $s = 0, 1, 2, \dots, m_1 - 1$. Considering sufficiently large values of k_1 in (3.6), we come to the conclusion that $a_s = b_s$ for all $s = 0, 1, \dots, m_1$, implying that

$$\begin{aligned} \gamma(m_1) &= \sum_{s=0}^{m_1-1} b_s m_1^{m_1-2-s} = \sum_{s=0}^{m_1-1} a_s m_1^{m_1-2-s} \\ &= \sum_{T \in \mathcal{T}(Q^*)} \sum_{f \in \Gamma(E(Q^*)-E(T))} m_1^{m_1-2-|f^{-1}(v_1)|} \prod_{i=2}^n k_i^{k_i-2-|f^{-1}(v_i)|}, \end{aligned}$$

implying that (3.1) holds for $k_1 = m_1$. Hence the result holds. \square

In the remainder of this section, we assume that $k_i \geq m_i + 1$ for all i with $1 \leq i \leq n$. Thus vertex v_{i,k_i} is not incident with any edge of M for each i . We will complete the proof of Theorem 3.1 by the approach explained in the two steps below:

(a) $\mathcal{ST}_Q(M)$ will be partitioned into $t(Q^*)2^{m-n+1}$ subsets denoted by $\Delta(T_0, f)$'s, corresponding to $t(Q^*)2^{m-n+1}$ ordered pairs (T_0, f) , where $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q^*) - E(T_0))$;

(b) then we show that for any given $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q^*) - E(T_0))$,

$$|\Delta(T_0, f)| = \prod_{i=1}^n k_i^{k_i-2-|f^{-1}(v_i)|}.$$

Step (a) above will be done by Algorithm B below which determines a spanning tree T_0 of Q^* and a mapping $f \in \Gamma(E(Q^*) - E(T_0))$ for any given $T \in \mathcal{ST}_Q(M)$.

Algorithm B ($T \in \mathcal{ST}_Q(M)$):

Step B1. Let T_n be T ;

Step B2. for $i = n, n-1, \dots, 1$, let $D_i = E_{T_i}(V_i) - \Phi(T_i, V_i, v_{i,k_i})$ and T_{i-1} be the graph obtained from T_i by deleting all edges in $D_i \cup E(T_i[V_i])$ and identifying all vertices of V_i as one, denoted by v_i , which is a vertex of Q^* ;

Step B3. output T_0 and f , where f is a mapping from $D_1 \cup D_2 \cup \dots \cup D_n$ to $V(Q^*)$ defined by $f(e) = v_i$ whenever $e \in D_i$.

By Lemma 2.1, each graph T_i produced in the process of running Algorithm B is indeed a tree and thus T_0 is a tree in $\mathcal{T}(Q^*)$. It is also clear that $D_1 \cup D_2 \cup \dots \cup D_n = E(Q^*) - E(T_0)$ and so the mapping f output by Algorithm B belongs to $\Gamma(E(Q^*) - E(T_0))$.

An example is presented below. Let T be a tree in $\mathcal{ST}_Q(M)$ as shown in Figure 6(a), where Q is a connected graph with a matching $M = \{e_1, e_2, \dots, e_8\}$ such that $Q - M$ has four components Q_1, Q_2, Q_3 and Q_4 isomorphic to complete graphs of orders 5, 4, 6, 5 respectively. If we run Algorithm B with this tree T as its input, then we have T_3, T_2, T_1 and T_0 as shown in Figure 6 and thus

$$D_4 = \{e_4\}, D_3 = \{e_1, e_2\}, D_2 = \{e_5, e_7\}, D_1 = \emptyset,$$

implying that the mapping $f \in \Gamma(E(Q^*) - E(T_0))$ output by Algorithm B, where $E(Q^*) - E(T_0) = \{e_1, e_2, e_4, e_5, e_7\}$, is the one given below:

$$f(e_1) = f(e_2) = v_3, f(e_4) = v_4, f(e_5) = f(e_7) = v_2.$$

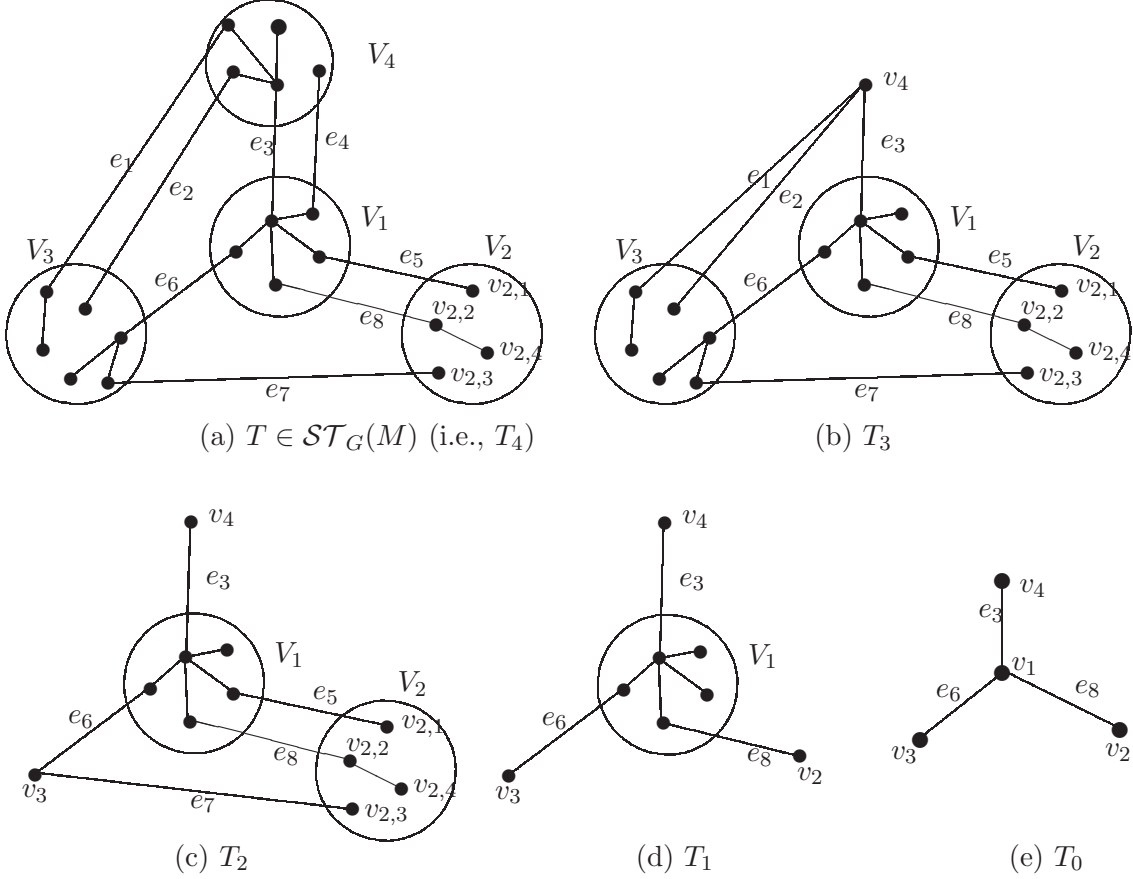


Figure 6: $T \in \mathcal{ST}_Q(M)$ (i.e., T_4) and T_3, T_2, T_1, T_0

Let ψ be a mapping from $\mathcal{ST}_Q(M)$ to the following set of ordered pair (T_0, f) 's:

$$\{(T_0, f) : T_0 \in \mathcal{T}(Q^*), f \in \Gamma(E(Q) - E(T_0))\},$$

defined by $\psi(T) = (T_0, f)$ if T_0 and f are output by running Algorithm B with input T . For any $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q) - E(T_0))$, let $\Delta(T_0, f) = \psi^{-1}(T_0, f)$. Thus $\mathcal{ST}_Q(M)$ is partitioned into $t(Q^*)2^{m-n+1}$ subsets $\Delta(T_0, f)$'s, where $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q) - E(T_0))$.

The proof of Theorem 3.1 now remains to determine the size of $\Delta(T_0, f)$ below.

Proposition 3.2 *For any $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q^*) - E(T_0))$, we have*

$$|\Delta(T_0, f)| = \prod_{i=1}^n k_i^{k_i-2-|f^{-1}(v_i)|}.$$

Proof. Let $D_i = f^{-1}(v_i) = \{e \in M - E(T_0) : f(e) = v_i\}$ for $i = 1, 2, \dots, n$. So $D_i \subseteq M_i$. By Algorithm B , T is a member of $\Delta(T_0, f)$ if and only if there exist trees T_1, T_2, \dots, T_{n-1} such that for $i = n, n-1, \dots, 1$, the following properties hold, where T_n is the tree T :

$$(P1) \quad V(T_i) = (V(T_{i-1}) - \{v_i\}) \cup V_i;$$

(P2) $T_i - V_i$ and $T_{i-1} - v_i$ are the same graph; and

(P3) $E_{T_{i-1}}(v_i) = \Phi(T_i, V_i, v_{i,k_i}) = E_{T_i}(V_i, V(T_i) - V_i) - D_i$ and $D_i \subseteq E_{T_i}(V_i, V(T_i) - V_i)$.

Let $U_i = \bigcup_{1 \leq j \leq i} V_j \cup \{v_{i+1}, \dots, v_n\}$. Observe that if properties (P1), (P2) and (P3) hold for all i with $1 \leq i \leq n$, then $V(T_i) = U_i$ for all $i = 0, 1, \dots, n$.

Now let $\Delta_0 = \{T_0\}$. Define sets $\Delta_1, \Delta_2, \dots, \Delta_n$ as follows. For $i = 1, 2, \dots, n$, let

$$\Delta_i = \bigcup_{T_{i-1} \in \Delta_{i-1}} \Psi(T_{i-1}),$$

where $\Psi(T_{i-1})$ is the set of all those spanning trees T_i of H_i such that properties (P1), (P2) and (P3) hold for T_i and T_{i-1} and H_i is the graph with $V(H_i) = U_i$ such that V_i is a clique of H_i , $H_i - V_i$ is the same as $T_{i-1} - v_i$ and $E_{H_i}(V_i) = E_{T_{i-1}}(v_i) \cup D_i$. Note that for each edge $e \in E_{H_i}(V_i)$, e is actually also an edge in Q and we assume that e joins the same pair of vertices as it does in Q unless e as an edge of Q has one end in some V_j with $j > i$, while in this case this end of e in H_i is v_j .

By (P1), (P2) and (P3), T_{i-1} is uniquely determined by any $T_i \in \Psi(T_{i-1})$. Thus $\Psi(T'_{i-1}) \cap \Psi(T''_{i-1}) = \emptyset$ for any distinct members T'_{i-1} and T''_{i-1} of Δ_{i-1} . For any $T_{i-1} \in \Delta_{i-1}$, observe that $\Psi(T_{i-1})$ is actually the set $\mathcal{ST}_{H_i}(V_i, T_{i-1}, D_i, v_{i,k_i})$, and thus by Proposition 2.3, we have

$$|\Psi(T_{i-1})| = k_i^{k_i-2-|D_i|}.$$

Hence $|\Delta_i| = k_i^{k_i-2-|D_i|} |\Delta_{i-1}|$ for all $i = 1, 2, \dots, n$. As $\Delta(T_0, f) = \Delta_n$, the result holds. \square

We end this section with a proof of Theorem 3.1.

Proof of Theorem 3.1: By Proposition 3.1, we may assume that $k_i > m_i$ for all $i = 1, 2, \dots, n$.

By the definition of ψ and $\Delta(T_0, f) = \psi^{-1}(T_0, f)$, we have

$$\mathcal{ST}_Q(M) = \bigcup_{\substack{T_0 \in \mathcal{T}(H) \\ f \in \Delta(E(H) - E(T_0))}} \Delta(T_0, f),$$

where the union gives a partition of $\mathcal{ST}_Q(M)$. Thus

$$|\mathcal{ST}_Q(M)| = \sum_{\substack{T_0 \in \mathcal{T}(H) \\ f \in \Gamma(E(H) - E(T_0))}} |\Delta(T_0, f)| = \sum_{\substack{T_0 \in \mathcal{T}(H) \\ f \in \Gamma(E(H) - E(T_0))}} \prod_{i=1}^n k_i^{k_i-2-|f^{-1}(v_i)|},$$

where the last step follows from Proposition 3.2. Hence Theorem 3.1 holds. \square

4 Proving Theorem 1.1 for $r \geq 1$

In this section, we shall prove Theorem 1.1 for the case $r \geq 1$.

For any graph G and edge e in G , let $G - e$ and G/e be the graphs obtained from G by deleting e and contracting e respectively. The following result is obvious.

Lemma 4.1 ([4, 5]) *For any graph G and edge e in G , we have*

$$t(G) = t(G - e) + t(G/e).$$

In particular, if e is a bridge of G , then $t(G) = t(G/e)$.

For any edge e in G , let $G_{\bullet e}$ be the graph obtained from G by inserting a vertex on e and G_{-e} be the graph obtained from $G - e$ by attaching a pendent edge to each end of e , as shown in Figure 7. Similarly, for any $E' \subseteq E(G)$, let $G_{\bullet E'}$ be the graph obtained from G by inserting a vertex on each edge of E' and $G_{-E'}$ be the graph obtained from $G - E'$ by attaching a pendent edge to each end of e for all $e \in E'$. Clearly $G_{\bullet E'}$ is the graph $S(G)$ when $E' = E(G)$.

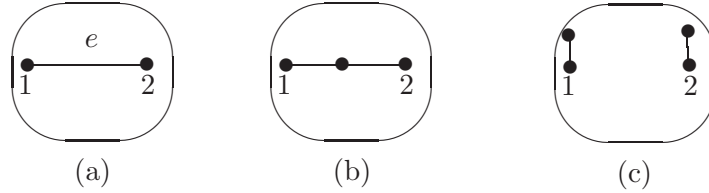


Figure 7: (a) G with edge e (b) The graph $G_{\bullet e}$ (c) The graph G_{-e}

By the definition of the line graph, the following lemma follows from Lemma 4.1.

Lemma 4.2 *Let G be any graph and e be an edge in G . Then*

$$t(L(G_{\bullet e})) = t(L(G)) + t(L(G_{-e})).$$

In particular, if e is a bridge of G , then $t(L(G_{\bullet e})) = t(L(G))$.

For any edge e in G and any non-negative integer r , let $G_{r\bullet e}$ be the graph obtained from G by inserting r new vertices on e , i.e., replacing e by a path of length $r + 1$ connecting the two ends of e . For any subset F of $E(G)$, let $G_{r\bullet F}$ be the graph obtained from G by replacing each edge e of F by a path of length $r + 1$ connecting the two ends of e .

Lemma 4.3 *Let G be any graph and F be any subset of $E(G)$. Then, for any $r \geq 0$,*

$$t(L(G_{r\bullet F})) = \sum_{E' \subseteq F} r^{|E'|} t(L(G_{-E'})). \quad (4.1)$$

Proof. Note that for any two vertices u, v in a graph H , if $N_H(u) = \{v\}$ and $d_H(v) = 2$, then $t(L(H)) = t(L(H - u))$. Thus, for any edge e of G and any positive integer r , by Lemma 4.2, we have

$$t(L(G_{r\bullet e})) = t(L(G_{(r-1)\bullet e})) + t(L(G_{-e})), \quad (4.2)$$

where $G_{0\bullet e}$ is G . Applying (4.2) repeatedly deduces that

$$t(L(G_{r\bullet e})) = t(L(G)) + rt(L(G_{-e})). \quad (4.3)$$

Note that (4.1) is obvious for $F = \emptyset$ or $r = 0$. Now assume that $e \in F$ and $r \geq 1$. By induction, we have

$$t(L(G_{r\bullet F - \{e\}})) = \sum_{E' \subseteq F - \{e\}} r^{|E'|} t(L(G_{-E'})). \quad (4.4)$$

By (4.3), we have

$$t(L(G_{r\bullet F})) = t(L(G_{r\bullet F - \{e\}})) + rt(L((G_{r\bullet F - \{e\}})_{-e})). \quad (4.5)$$

Thus (4.1) follows immediately from (4.4). \square

We are now ready to prove Theorem 1.1 for the case $r \geq 1$.

Proof of Theorem 1.1 for $r \geq 1$: Assume that $r \geq 1$. By Lemma 4.3, we have

$$t(L(S_r(G))) = \sum_{E' \subseteq E(G)} r^{|E'|} t(L(G_{-E'})). \quad (4.6)$$

The above summation needs only to take those subsets E' of $E(G)$ with $t(L(G_{-E'})) > 0$ (i.e. $G - E'$ is connected). Now let E' be any fixed subset of $E(G)$ such that $G - E'$ is connected and let H denote $G_{-E'}$. By Theorem 1.1 for $r = 0$ (i.e., (1.9)),

$$t(L(G_{-E'})) = \sum_{T' \in \mathcal{T}(H)} \sum_{g \in \Gamma(E(H) - E(T'))} \prod_{v \in V(H)} d_H(v)^{d_H(v) - 2 - |g^{-1}(v)|}. \quad (4.7)$$

Observe that $V(G) \subseteq V(H)$. For any $v \in V(H)$, if $v \in V(G)$, then $d_H(v) = d_G(v)$; otherwise, $d_H(v) = 1$. Thus

$$\prod_{v \in V(H)} d_H(v)^{d_H(v) - 2 - |g^{-1}(v)|} = \prod_{v \in V(G)} d_G(v)^{d_G(v) - 2 - |g^{-1}(v)|}. \quad (4.8)$$

For each $T' \in \mathcal{T}(H)$, T' contains all pendent edges in H and so T' corresponds to T , where $T = T'[V(G)]$, which is a spanning tree of $G - E'$. Thus $E(H) - E(T') = E(G - E') - E(T)$ and

$$t(L(G_{-E'})) = \sum_{T \in \mathcal{T}(G - E')} \sum_{g \in \Gamma(E(G - E') - E(T))} \prod_{v \in V(G)} d_G(v)^{d_G(v) - 2 - |g^{-1}(v)|}. \quad (4.9)$$

By (4.6) and (4.9),

$$t(L(S_r(G))) = \sum_{E' \subseteq E(G)} r^{|E'|} \sum_{T \in \mathcal{T}(G - E')} \sum_{g \in \Gamma(E(G - E') - E(T))} \prod_{v \in V(G)} d_G(v)^{d_G(v) - 2 - |g^{-1}(v)|}. \quad (4.10)$$

By replacing $E(G) - E' - E(T)$ by E'' , (4.10) implies that

$$\begin{aligned}
t(L(S_r(G))) &= \sum_{E'' \subseteq E(G)} \sum_{T' \in \mathcal{T}(G-E'')} \sum_{g \in \Gamma(E'')} r^{|E(G)|-|E''|-|E(T')|} \prod_{v \in V(G)} d_G(v)^{d_G(v)-2-|g^{-1}(v)|} \\
&= \sum_{E'' \subseteq E(G)} r^{|E(G)|-|E''|-|V(G)|+1} t(G-E'') \sum_{g \in \Gamma(E'')} \prod_{v \in V(G)} d_G(v)^{d_G(v)-2-|g^{-1}(v)|} \\
&= \sum_{E''' \subseteq E(G)} r^{|E'''|-|V(G)|+1} t(G[E''']) \sum_{g \in \Gamma(E-E''')} \prod_{v \in V(G)} d_G(v)^{d_G(v)-2-|g^{-1}(v)|}.
\end{aligned}$$

Hence the case $r \geq 1$ of Theorem 1.1 holds. \square

5 Proof of Conjecture 1.1

Now we turn back to those connected graphs G mentioned in Conjecture 1.1 and apply the following result and Theorem 1.1 to deduce a relation between $t(L(S_r(G)))$ and $t(G)$. The case $r = 1$ of this relation is exactly the conclusion of Conjecture 1.1.

Lemma 5.1 *Let H be any connected graph of order n and size m . For any integer i with $0 \leq i \leq m - n + 1$, we have*

$$\binom{m-n+1}{i} t(H) = \sum_{\substack{E' \subseteq E(H) \\ |E'|=i}} t(H-E').$$

Proof. We prove this result by providing two different methods to determining the size of the following set:

$$\Theta = \{(T, E') : T \text{ is a spanning tree of } H \text{ and } E' \subseteq E(H) - E(T) \text{ with } |E'| = i\}.$$

Note that for each spanning tree T of H , as $|E(H)| = m$ and $|E(T)| = n - 1$, the number of subsets E' of $E(H) - E(T)$ with $|E'| = i$ is $\binom{m-n+1}{i}$. On the other hand, for each $E' \subseteq E(H)$ with $|E'| = i$, there are exactly $t(H - E')$ spanning trees T of G such that $E' \subseteq E(H) - E(T)$. Thus the result holds. \square

We now deduce the following consequence of Theorem 1.1 for those connected graphs G mentioned in Conjecture 1.1.

Corollary 5.1 *Let G be a connected graph of order $n + s$ and size $m + s$ in which s vertices are of degree 1 and all others are of degree k , where $k \geq 2$. Then, for any $r \geq 0$,*

$$t(L(S_r(G))) = k^{m+s-n-1} (rk + 2)^{m-n+1} t(G).$$

Proof. For any $E' \subseteq E(G)$ with $t(G[E']) \neq 0$, E' contains every bridge of G , and so $d(u_e) = d(v_e) = k$ for all $e \in E(G) - E'$. By Theorem 1.1, we have

$$\begin{aligned}
t(L(S_r(G))) &= (k^{k-2})^n \sum_{E' \subseteq E(G)} t(G[E']) r^{|E'|-(n+s)+1} (2k^{-1})^{(m+s)-|E'|} \\
&= (k^{k-2})^n r^{-(n+s)+1} (2k^{-1})^{(m+s)} \sum_{E' \subseteq E(G)} t(G[E']) r^{|E'|} (2k^{-1})^{-|E'|} \\
&= (k^{k-2})^n r^{-(n+s)+1} (2k^{-1})^{(m+s)} \sum_{E'' \subseteq E(G)} t(G - E'') r^{|E(G)|-|E''|} (2k^{-1})^{|E''|-|E(G)|} \\
&= (k^{k-2})^n r^{-(n+s)+1} (2k^{-1})^{(m+s)} \sum_{j=0}^{m-n+1} r^{m+s-j} (2k^{-1})^{j-m-s} \sum_{\substack{E'' \subseteq E(G) \\ |E''|=j}} t(G - E'') \\
&= (k^{k-2})^n \sum_{j=0}^{m-n+1} r^{m-n+1-j} (2k^{-1})^j \binom{m-n+1}{j} t(G) \quad (\text{by Lemma 5.1}) \\
&= (k^{k-2})^n (r + 2k^{-1})^{m-n+1} t(G) \\
&= k^{n(k-2)-(m-n+1)} (kr + 2)^{m-n+1} t(G) \\
&= k^{m+s-n-1} (kr + 2)^{m-n+1} t(G),
\end{aligned}$$

where the last expression follows from the equality $2(m+s) = kn + s$ by the given conditions on G . Hence the result is obtained. \square

Notice that (1.3) is the special case of Corollary 5.1 for $r = 0$ while the conclusion of Conjecture 1.1 is the special case of Corollary 5.1 for $r = 1$.

We end this section with the the following result on some special bipartite graphs, which can be obtained by applying Lemma 5.1 and the case $r = 0$ of Theorem 1.1.

Corollary 5.2 *Let $G = (A, B; E)$ be a connected bipartite graph of order n and size m such that $d(x) \in \{1, a\}$ for all $x \in A$ and $d(y) \in \{1, b\}$ for all $y \in B$, where $a \geq 2$ and $b \geq 2$. Then*

$$t(L(G)) = a^{(a-2)n_1} b^{(b-2)n_2} (a^{-1} + b^{-1})^{m-n+1} t(G),$$

where n_1 is the number of vertices x in A with $d(x) = a$ and n_2 is the number of vertices y in B with $d(y) = b$.

The result of Corollary 5.2 in the case that G is an (a, b) -semiregular bipartite graph was originally due to Cvetković (see Theorem 3.9 in [13], §5.2 of [15], or [17]).

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